Lecture 12: Predictive maps

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- This "predictive map" may be represented in the hippocampus and updated by dopaminergic temporal difference errors or a temporally asymmetric learning rule.
- A low-dimensional representation in entorhinal cortex may regularize learning toward predictions that are smooth across the state space.

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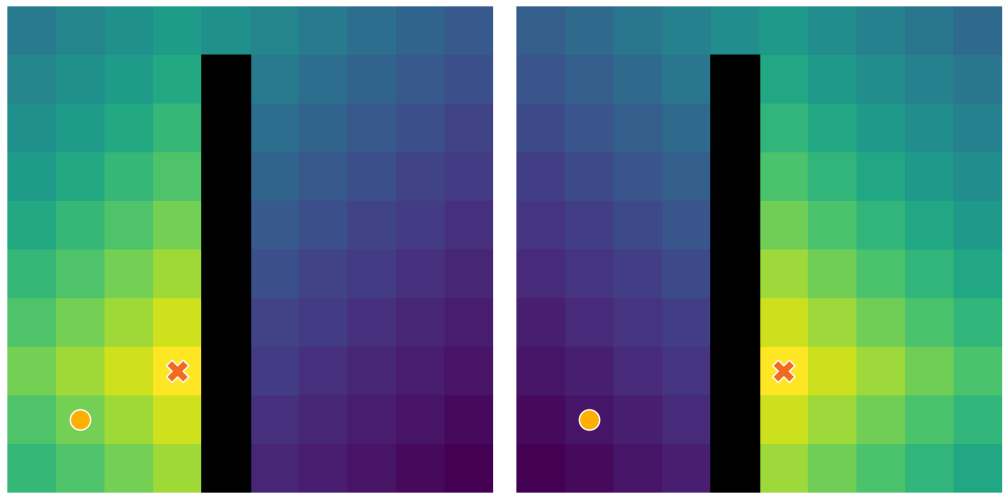
- Simplest case: value functions and policies are stored in look-up tables, requiring no computation beyond finding the right entry for a given state-action pair.
- More complex case: functions are neural networks, still relatively cheap.
- The trade-off is that these representations are relatively inflexible: if the world changes, the representations may need to be relearned.

Local changes have non-local effects on the value function

The goal is shown as an X and the agent is shown as a circle. Black squares are walls. The shading of the other squares shows the value function. The two gridworlds differ only by the goal placement.



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- States may need to be resampled in order to update the value function.
- When experience is scarce, relearning may be prohibitively costly.
- But animals don't always need to relearn. They're sometimes capable of rapid adaptation. How?

Latent learning

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- Contextual fear conditioning: shock paired with a context. If the shock is delivered immediately, conditioning is *weaker* compared to the same procedure with a delayed shock [Fanselow 1990].
- Additional time is required for the animal to explore the box and form a predictive map, which then allows the shock delivery to propagate across the whole state space. Our goal is to formalize this idea.

The successor representation

- The value function admits the following decomposition:

$$V^{\pi}(s) = \sum_{\tilde{s}} M^{\pi}(s, \tilde{s}) R(\tilde{s})$$

where

$$M^{\pi}(s, \tilde{s}) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^{t} \mathbb{I}[s_t = \tilde{s}] | s_0 = s, \pi\right]$$

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  is the SR (expected discounted number of times an agent visits state s' on a trajectory starting in state s).
- The SR formalizes our intuitions about a predictive map: it specifies where the agent will be in the near future (with a predictive horizon set by the discount factor γ).

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- Given the SR, values can be computed by a linear operation on the reward function.
- Key feature: separation of state prediction and reward prediction. This allows the model to flexibly reuse state predictions when the reward changes.

Bellman equation for the successor representation

- Analogous to the value function:

$$M^{\pi}(s, \tilde{s}) = \mathbb{I}[s = \tilde{s}] + \gamma \sum_{s'} T^{\pi}(s, s') M^{\pi}(s', \tilde{s})$$

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- TD learning can be used to update an estimate of the SR, $\hat{M}^{\pi}$, from observed state transitions ($s \to s'$):

$$\Delta \hat{M}^{\pi}(s, \tilde{s}) \propto \mathbb{I}[s = s'] + \gamma \hat{M}^{\pi}(s', \tilde{s}) - \hat{M}^{\pi}(s, \tilde{s})$$

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- The immediate rewards can also be learned by a simple error-driven update after observing reward r in state s:

$$\Delta \hat{R}(s) \propto r - \hat{R}(s)$$

Illustration with contextual fear conditioning

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- Eliminating pre-exposure attenuates conditioned fear to the context: upon return to the context, the animal is likely placed in a location different from the one that coincided with the shock, so it will only produce a fear response if the shock expectation has been propagated across the state space.

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- Compared to model-free algorithms, SR can more flexibly adapt to changes in the reward function.
- Model-based algorithms are maximally flexible, since they can in principle adapt to any change in the reward and transition functions, but at the cost of higher computational complexity.

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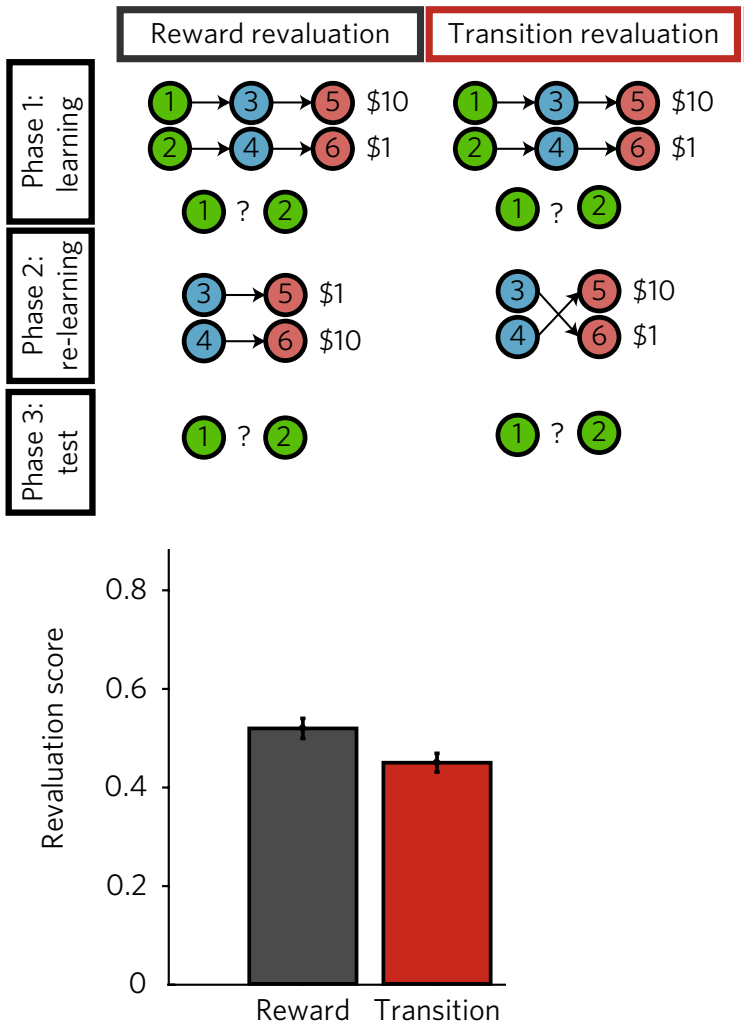
- Changes to the reward function induce local changes to the reward function estimate.
- Model-based algorithms are equally sensitive to both transition and reward function changes, whereas model-free algorithms are equally insensitive.

Transition and reward revaluation

(Top) Experimental design. Circles denote states and arrows denote transitions. (Bottom) Behavioral results. Revaluation score denotes the change in preference rating after versus before the relearning phase.



[Momennejad et al 2017]

A predictive map in the hippocampus

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- Place fields concentrate near rewards [Hollup et al 2001].
- Lesions of the hippocampus impair several forms of latent learning, and stimulus-stimulus predictions are encoded in hippocampal activity during latent learning.

A predictive map in the hippocampus

- SR offers a different account of these phenomena. Assume that each state corresponds to a location, and the place field for a single cell corresponds to one column of the SR estimate, $\hat{M}^{\pi}(\cdot, \tilde{s})$; we will refer to this as the *SR field* tuned to location $\tilde{s}$.

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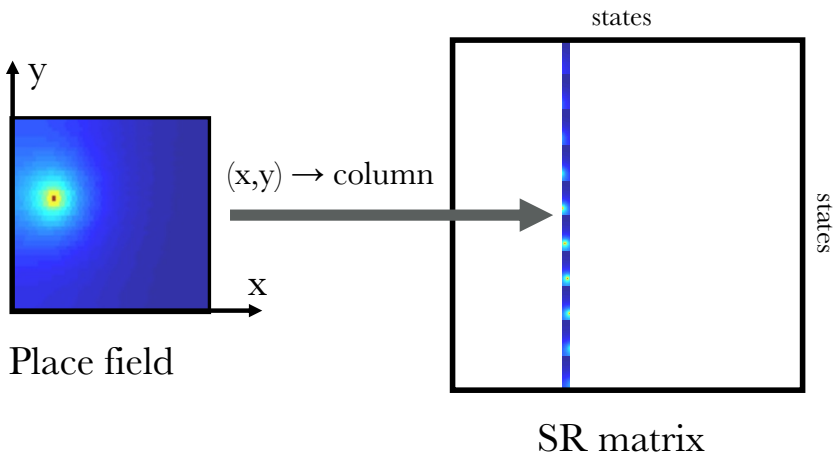
- An SR field is a *retrospective representation* in the sense that activity is maximal for states leading to $\tilde{s}$.
- A row of the SR, $\hat{M}^{\pi}(s, \cdot)$, corresponds to the population code for state s. This is a *predictive representation* in the sense that activity is maximal for cells tuned to states leading from s.

From place fields to SR fields

Place field corresponding to a single column of the SR



[Gershman 2018]

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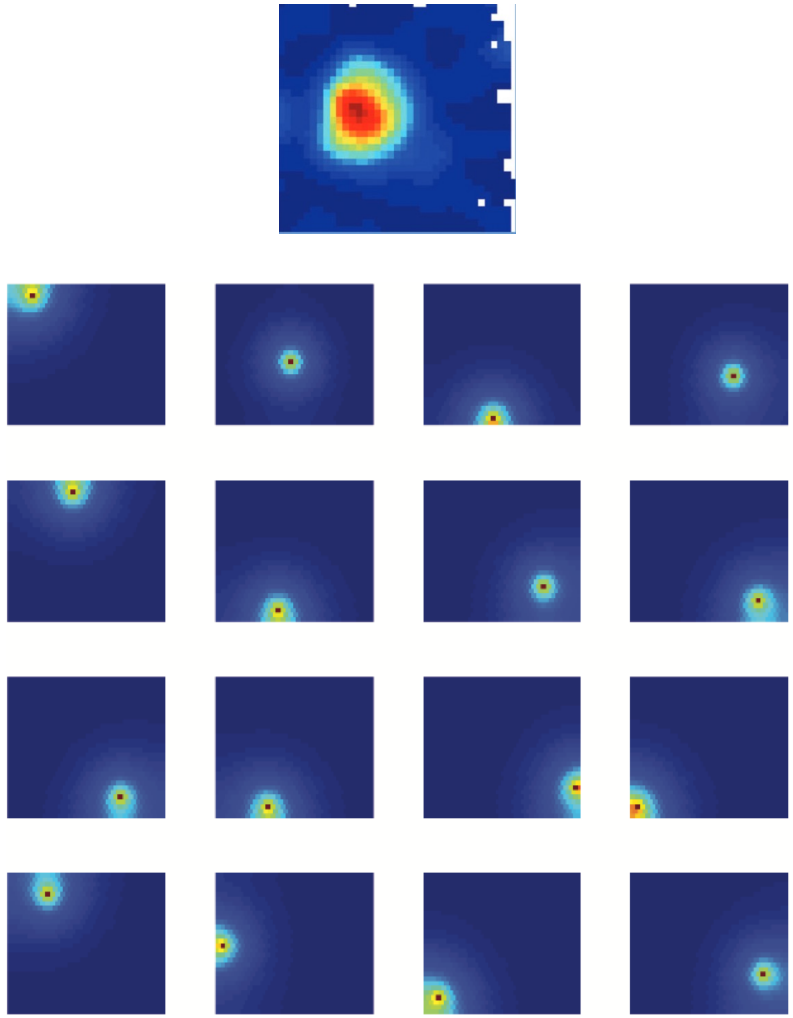
- If rats are instead trained on the linear track or circular water maze, systematic asymmetries in the direction of travel will bias the SR fields backwards in the direction of travel (reflecting the retrospective nature of the representation) and concentrate them near the platform (reflecting the shift in tuning toward frequently visited locations).
- Thus, the SR can explain both classical and non-classical aspects of place fields.

From place fields to SR fields

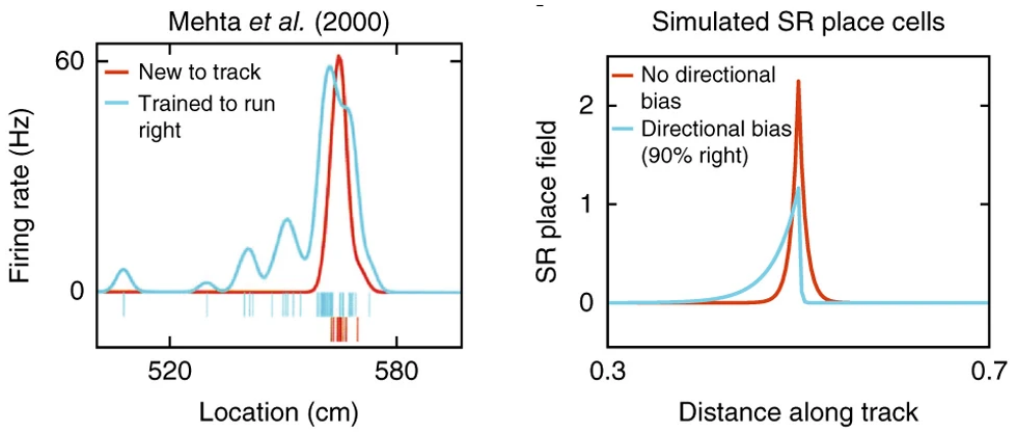
Real and simulated place fields in a random foraging paradigm



[Moser et al 2015; Stachenfeld et al 2017]

From place fields to SR fields

(Left) Real neurons. (Right) Successor representation.



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- This **generalized prediction error** hypothesis (generalizing from scalar rewards to feature vectors) can resolve a number of puzzles concerning dopamine.

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- If states correspond to reward identities, then generalized prediction errors can capture the sensitivity of dopamine neurons to sensory surprise.
- If dopamine neurons are tuned to different states, then reward identity will be encoded at the population level as long as the generalized prediction errors are non-zero.

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- Evidence is suggestive but indirect. Over-representation of high-reward locations in the hippocampus can be attenuated by inhibition of dopamine inputs to the hippocampus [Mamad et al 2017; Krishnan et al 2022].
- Hippocampal plasticity depends on dopamine signaling [Tsetsenis et al 2023].

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Recurrent neural network implementation

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- Let x_t denote an input to a linear recurrent neural network (RNN) with recurrent synaptic connectivity matrix Ω, where Ω_{ij} denotes the strength of the synapse connecting neuron i to neuron j.
- Firing rate dynamics of the recurrent units (z) can be written in discrete time as:

$$z_{t+1} = x_t + \gamma z_t \Omega$$

where the discount factor γ plays the role of a synaptic gain.

Recurrent neural network implementation

- If $x_t = x$ is a constant one-hot encoding of the state (i.e., $x(s) = 1$ whenever the system is in state s, otherwise $x(s) = 0$), and the connectivity matrix corresponds to the state transition matrix, $\Omega = T^\pi$, then at steady state the recurrent units will converge to the row of the SR corresponding to x:

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- Intuitively, the RNN is progressing the predictions forward at each iteration and adding up the results.

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- Update rule uses only information local to a synapse, and converges to the transition matrix asymptotically.

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- If expected rewards can be represented as a linear function of the features, $R(s) = \sum_d w_d x_d$, then the value function can be represented as a linear function of **successor features** (SFs), $\psi^\pi(s)$:

$$V^\pi(s) = \sum_d w_d \psi^\pi_d(s), \qquad \psi^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t f(s_t) | s_0 = s, \pi\right]$$

where $x = f(s)$.

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- If different subpopulations of dopamine neurons signal TD errors for different features, this model may explain how a single canonical computation (the TD update) is compatible with the diversity of feature sensitivity across dopamine neurons [Engelhard et al 2019].

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- The simplest approach to propagating this value information across all states and actions is to sample random state-action pairs (s, a) and use the Bellman equation to determine the updated value (Bellman backup):

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- Repeating this update for different state-action pairs is guaranteed to converge to the correct values under the optimal policy π^*.

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- Two drawbacks: (1) requires a model (T and R), which might not be available; (2) might be very inefficient or intractable if the state/action space is vast.
- Instead of using a model, use a stochastic approximation based on samples $e = (s, a, s', r)$ drawn from memory:

$$\Delta \hat{Q}^{\pi}(s, a) \propto r + \gamma \max_{a'} \hat{Q}^{\pi}(s', a') - \hat{Q}^{\pi}(s, a)$$

Prioritized replay for offline updating

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- Sampling from memory also has the advantage that it focuses updating on parts of the state-action space that the agent is likely to visit (assuming the past is representative of the future).

Prioritized replay for offline updating

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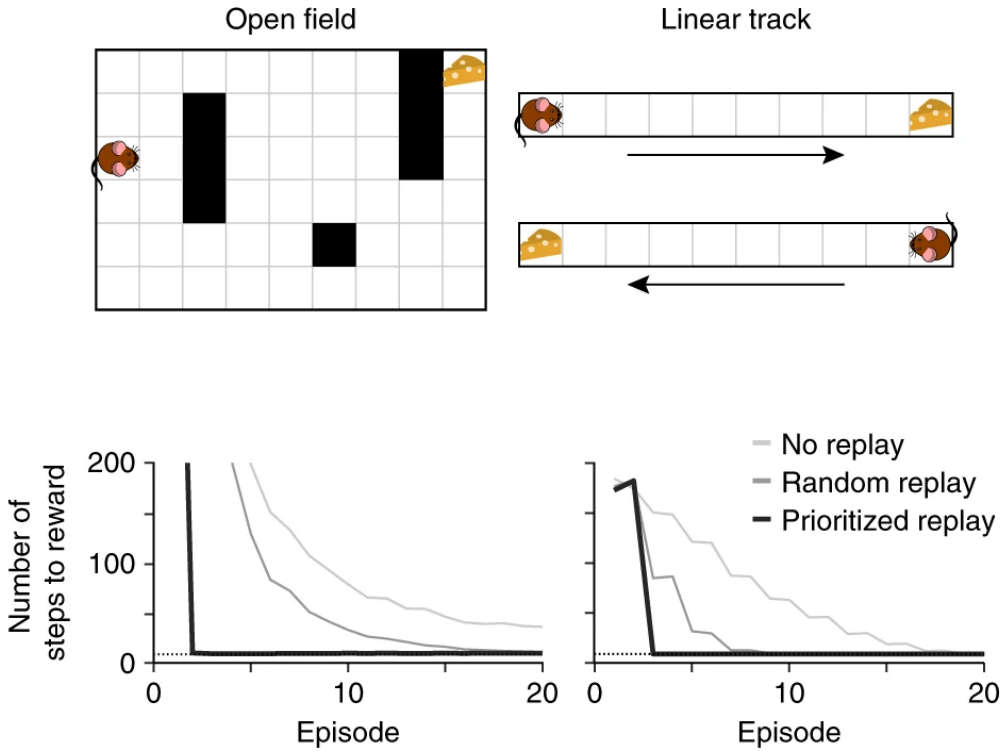
Prioritized replay for offline updating

- The way memory is sampled can have a big impact on the speed of learning.
- Optimal prioritization strategy can do much better than uniformly sampling memory.

Optimal vs. uniform memory sampling



[Mattar & Daw 2018]

Prioritized replay for offline updating

- Optimal prioritization strategy is defined by the *expected value of backup* (EVB):

$$\text{EVB}(k) = \hat{V}^{\pi_k}(s) - \hat{V}^{\pi}(s)$$

where s is the agent's current state, k indexes memories $\{e_k\}$, and π_k is the new policy after applying a backup to the retrieved memory.

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- EVB quantifies the improvement in the policy that could be attained by applying a backup to a particular memory.
- Optimal prioritization strategy schedules these updates based on decreasing EVB.

Prioritized replay for offline updating

- EVB can be expressed in the following way:

$$\text{EVB}(s, k) = \text{Gain}(k) \cdot \text{Need}(s, k),$$

where $\text{Gain}(k) = V^{\pi_k}(s_k) - V^{\pi}(s_k)$ is the improvement local to the retrieved state, and $\text{Need}(s, k) = M^{\pi}(s, s_k)$ is how often the memory is needed (how often the retrieved state will be revisited in the future), quantified by the SR.

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- Thus, in this setting the predictive map serves to guide memory retrieval.

Prioritized replay for offline updating

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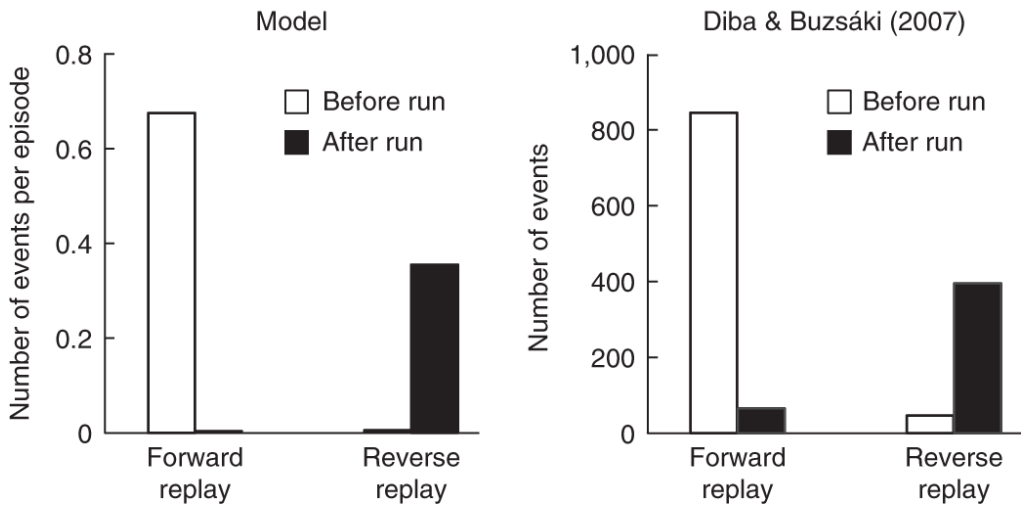
- If the agent encounters an unexpected reward, the gain term will be greatest for the current state's predecessors, leading to backward replay.
- In the absence of a prediction error, the need term dominates, leading to forward replay (states expected in the near future have higher need). This is why forward replay tends to be observed before an animal starts a run on the linear track.

Frequency of forward vs. reverse replays



[Mattar & Daw 2018]

Study question

How might you incorporate predictive maps into the kind of policy gradient algorithms described in the last lecture?

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- Predictive maps enable the brain to (partially) escape the inflexibility that curses model-free algorithms, while achieving comparable efficiency.

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- A canonical RL computation (TD learning) can be generalized to construct predictive maps.
- Hippocampus appears to represent a predictive map, possibly updated by vector-valued dopamine signals encoding generalized prediction errors.